Q1 Let $f(x) = x, \quad 0 \le x \le 1, \quad P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ be a partition of $[0,1]$. Compute $U(P,f)$ & $L(P,f)$

Soln Subintervals of P are

$$[0, \tfrac{1}{4}], \ [\tfrac{1}{4}, \tfrac{1}{2}], \ [\tfrac{1}{2}, \tfrac{3}{4}], \ [\tfrac{3}{4}, 1]$$

here $m_1 = 0, \quad m_2 = \frac{1}{4}, \quad m_3 = \frac{1}{2}, \quad m_4 = \frac{3}{4}$

$M_1 = \frac{1}{4}, \quad M_2 = \frac{1}{2}, \quad M_3 = \frac{3}{4}, \quad M_4 = 1$

$\Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x_4 = \frac{1}{4}$

$$U(P,f) = \sum_{r=1}^{4} M_r \Delta x_r = \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + 1\cdot\frac{1}{4}$$

$$= \frac{5}{8}$$

$$L(P,f) = \sum_{r=1}^{4} m_r \Delta x_r$$

$$= 0\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} = \frac{3}{8}$$

Q2 Show that a constant function K is R integrable and $\int_a^b K\,dx = K(b-a)$

Soln For any partition P of the interval $[a,b]$

$$L(P,f) = \sum_{r=0}^{n} m_r \Delta x_r = K[x_1 - x_0 + x_2 - x_1 + \cdots + x_n - x_{n-1}]$$

$$= K[x_n - x_0] = K(b-a)$$

$$\therefore \int_a^b K\,dx = \sup L(P,f) = K(b-a)$$

also $U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r$

$= K(x_1 - x_0 + x_2 - x_1 + \cdots + x_n - x_{n-1}) = K(x_n - x_0)$

$= K(b-a)$

$\therefore \overline{\int_a^b} K\,dx = \inf U(P, f) = K(b-a)$

Thus $\underline{\int_a^b} K\,dx = \overline{\int_a^b} K\,dx = \int_a^b K\,dx$

So $K \in R[a,b]$ and $\int_a^b K\,dx = K(b-a)$

Q3 Let $f(x) = x$ on $[0,1]$. Compute $\overline{\int_0^1} x\,dx$ and $\underline{\int_0^1} x\,dx$ by dividing $[0,1]$ into n equal parts and hence show f is R integrable

Soln Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{r-1}{n}, \frac{r}{n}, \ldots \frac{n}{n} = 1\right\}$

Then $m_r = \frac{r-1}{n}$, $M_r = \frac{r}{n}$ & $\Delta x_r = \frac{1}{n}$

for $r = 1, 2, \ldots n$

$$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \left(\frac{r-1}{n}\right) \cdot \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{r=1}^{n} (r-1)$$

$$= \frac{1}{n^2} \left[1 + 2 + \cdots + (n-1)\right]$$

$$= \frac{1}{n^2} \frac{(n-1)n}{2} = \frac{n-1}{2n}$$

$$U(P,f) = \sum_{r=1}^{n} M_r \, \Delta x_r$$

$$= \sum_{r=1}^{n} \frac{r}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{r=1}^{n} r = \frac{1}{n^2}(1+2+\dots+n)$$

$$= \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}$$

$$\underline{\int_0^1} f(x)\,dx = \lim_{\|P\| \to 0} L(P,f) = \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$$

and $\overline{\int_0^1} f(x)\,dx = \lim_{\|P\|\to 0} U(P,f) = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$

$\underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx$, where $f(x) = x$

so $f(x) = x \in R[0,1]$ and $\int_0^1 x\,dx = \frac{1}{2}$

Q4 Let $f(x) = x^2$ on $[0,a]$, $a>0$, show that $f \in R[0,a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$

Sol $P = \left\{\frac{ra}{n},\ r = 0,1,\dots n\right\}$ be partition of $[0,a]$

then $m_r = \frac{(r-1)^2 a^2}{n^2}$, $M_r = \frac{r^2 a^2}{n^2}$ and $\Delta x_r = \frac{a}{n}$

$$L(P,f) = \sum_{r=1}^{n} m_r \, \Delta x_r = \sum_{r=1}^{n} \frac{(r-1)^2 a^2}{n^2} \cdot \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} (r-1)^2 = \frac{a^3}{n^3} \cdot \frac{(n-1)n(2n-1)}{6}$$

$$= \frac{a^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r$$

$$= \sum_{r=1}^{n} \frac{r^2 a^2}{n^2} \cdot \frac{a}{n} = \frac{a^3}{n^3} \sum_{r=1}^{n} r^2$$

$$= \frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$\underline{\int_0^a} f(x)\,dx = \lim_{\|P\| \to 0} L(P,f)$$

$$= \lim_{n \to \infty} \frac{a^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) = \frac{a^3}{3}$$

and $\overline{\int_0^a} f(x)\,dx = \lim_{\|P\| \to 0} U(P,f)$

$$= \lim_{n \to \infty} \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{a^3}{3}$$

Thus $\underline{\int_0^a} f(x)\,dx = \overline{\int_0^a} f(x)\,dx$

$f \in R[0,a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$

Q8 Show that the function

$$f(x) = \begin{cases} \sqrt{1-x^2}, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$$

is not R integrable on $[0,1]$

Soln $(\sqrt{1-x^2})^2 - (1-x)^2 = (1-x^2) - (1-2x+x^2)$

$= 2x(1-x) > 0 \ \forall \ x \in (0,1)$

$\therefore \sqrt{1-x^2} > 1-x$, for $0 < x < 1$

Let $I_r = [x_{r-1}, x_r]$ be subinterval

$M_r = \sqrt{1-x^2}$ and $m_r = 1-x$

$$\underline{\int_0^1} f(x)\,dx = \int_0^1 (1-x)\,dx = \frac{1}{2}$$

$$\overline{\int_0^1} f(x)\,dx = \int_0^1 \sqrt{1-x^2}\,dx = \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= \frac{\pi}{4}$$

$$\underline{\int_0^1} f(x)\,dx \neq \overline{\int_0^1} f(x)\,dx$$

hence $f \notin R[0,1]$